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Quasi-Poisson actions and massive non-rotating BTZ black holes

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Abstract

Using ideas from an article of Bieliavsky, Rooman and Spindel on BTZ black holes, I construct a family of interesting examples of quasi-Poisson actions as defined by Alekseev and Kosmann-Schwarzbach. As an application, I obtain a genuine Poisson structure on $SL(2, \mathbb{R})$ which induces a Poisson structure on a BTZ black hole.

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1. Introduction

In [4], Bieliavsky et al. construct a Poisson structure on massive non-rotating BTZ black holes; in [3], Bieliavsky et al. construct a star product on the same black hole. The direction

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of this deformation is a Poisson bivector field which has the same symplectic leaves as the Poisson bivector field of [4]: roughly speaking, they correspond to orbits under a certain twisted action by conjugation.

In the present paper, I wish to show how techniques used in [4] in conjunction with techniques of the theory of quasi-Poisson manifolds (see [1,2]) can be used to construct an interesting family of manifolds with a quasi-Poisson action and how a particular case of this family leads to a genuine Poisson structure on a massive non-rotating BTZ black hole with similar symplectic leaves as in [4,3].

2. Main results

I will not recall here the basic definitions in the theory of quasi-Poisson manifolds and quasi-Poisson actions. The reader will find these definitions in Alekseev and Kosmann-Schwarzbach [1], and in Alekseev et al. [2].

Let G be a connected Lie group of dimension n and $\mathfrak{g}$ its Lie algebra, on which G acts by the adjoint action Ad. Assume we are given an Ad-invariant non-degenerate bilinear form K on $\mathfrak{g}$. For example, if G is semi-simple, then K could be the Killing form. In the following, I will denote by K again the linear isomorphism

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{g}^* \\ x &\longmapsto K(x, \cdot). \end{aligned}$$

Let $D = G \times G$ and $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ its Lie algebra. Define an Ad-invariant non-degenerate bilinear form $\langle\, , \rangle$ of signature (n, n) by

$$\begin{aligned} \mathfrak{d} \times \mathfrak{d} = (\mathfrak{g} \oplus \mathfrak{g}) \times (\mathfrak{g} \oplus \mathfrak{g}) &\longrightarrow \mathbb{R} \\ ((x, y), (x', y')) &\longmapsto K(x, x') - K(y, y'). \end{aligned}$$

Assume there is an involution σ on G which induces an orthogonal involutive morphism, again denoted by σ, on $\mathfrak{g}$. Let $\Delta_+ : G \to D$ and $\Delta_+^\sigma : G \to D$ be given by

$$\Delta_+(g) = (g, g)$$

and

$$\Delta_+^\sigma(g) = (g, \sigma(g)).$$

Denote by G_+ and G_+^σ their respective images in D. Let $S = D/G_+$ and $S^\sigma = D/G_+^\sigma$. Then both S and S^σ are isomorphic to G. The isomorphism between S and G is induced by the map

$$\begin{aligned} D &\longrightarrow G \\ (g, h) &\longmapsto gh^{-1}, \end{aligned}$$

whereas the isomorphism between S^σ and G is induced by

$$\begin{aligned} D &\longrightarrow G \\ (g,h) &\longmapsto g\sigma(h)^{-1}. \end{aligned}$$

I will use these two isomorphisms to identify S and G, and S^σ and G. Denote again by $\Delta_+ : \mathfrak{g} \to \mathfrak{d}$ and $\Delta_+^\sigma : \mathfrak{g} \to \mathfrak{d}$ the morphisms induced by $\Delta_+ : G \to D$ and $\Delta_+^\sigma : G \to D$ respectively. Let $\Delta_- : \mathfrak{g} \to \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ and $\Delta_-^\sigma : \mathfrak{g} \to \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ be defined by

$$\Delta_-(x) = (x, -x),$$

and

$$\Delta_-^\sigma(x) = (x, -\sigma(x)).$$

Let $\mathfrak{g}_- = \mathrm{Im}(\Delta_-)$ and $\mathfrak{g}_-^\sigma = \mathrm{Im}(\Delta_-^\sigma)$. We have two quasi-triples $(D, G_+, \mathfrak{g}_-)$ and $(D, G_+^\sigma, \mathfrak{g}_-^\sigma)$. They induce two structures of quasi-Poisson Lie group on D, of respective bivector fields P_D and P_D^σ, and two structures of quasi-Poisson Lie group on G_+ and G_+^σ of respective bivector fields P_{G_+} and $P_{G_+^\sigma}$. I will simply write G_+, respectively G_+^σ, to denote the group together with its quasi-Poisson structure. Of course, these quasi-Poisson structures are pairwise isomorphic. More precisely, the isomorphism $\mathrm{Id} \times \sigma : (g,h) \longmapsto (g, \sigma(h))$ of D sends P_D on P_D^σ and vice-versa. This isomorphism can be used to deduce some of the results given at the beginning of the present article from the results of Alekseev and Kosmann-Schwarzbach [1]; but it takes just as long to redo the computations, and that is what I do here.

According to [1], the bivector field P_D, respectively P_D^σ, is projectable onto S, respectively S^σ. Let P_S and $P_{S^\sigma}^\sigma$ be their respective projections. Using the identifications between S and G, and S^σ and G, one can check that P_S and $P_{S^\sigma}^\sigma$ are the same bivector fields on G. What is more interesting, and what I will prove, is the following Theorem.

Theorem 2.1. *The bivector field P_D^σ is projectable onto S. Let P_S^σ be its projection. Identify S with G and trivialise their tangent space using right translations, then for s in S and ξ in $\mathfrak{g}^* \simeq T_s^*S$ there is the following explicit formula*

$$P_S^\sigma(s)(\xi) = \tfrac{1}{2}(\mathrm{Ad}_{\sigma(s)^{-1}} - \mathrm{Ad}_s) \circ \sigma \circ K^{-1}(\xi).$$

Moreover, the action

$$\begin{aligned} G_+^\sigma \times S &\longrightarrow S \\ (g,s) &\longmapsto gs\sigma(g)^{-1} \end{aligned} \tag{1}$$

of G_+^σ on (S, P_S^σ) is quasi-Poisson in the sense of Alekseev and Kosmann-Schwarzbach [1]. The image of $P_S^\sigma(s)$, seen as a map $T_s^*S \longrightarrow T_sS$, is tangent to the orbit through s of the action of G^σ on *S.*

In the setting of the above Theorem, the bivector field P_S^σ is G^σ invariant; hence if F is a subgroup of G^σ and $\mathbf{I}$ is an F-invariant open subset of S such that the action of F on $\mathbf{I}$ is principal then $F\backslash\mathbf{I}$ is a smooth manifold and P_S^σ descends to a bivector field on it. An application of this remark is the following Theorem.

Theorem 2.2. *Let* $G = \mathrm{SL}(2, \mathbb{R})$. *Let*

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and choose $\sigma = \mathrm{Ad}_H$. *Let*

$$\mathbf{I} = \left\{ \begin{bmatrix} u+x & y+t \\ y-t & u-x \end{bmatrix} | \mathbf{u^2 - x^2 - y^2 + t^2 = 1, t^2 - y^2 > 0} \right\}$$

be an open subset of S. *Let* F *be the following subgroup of* G

$$F = \{\exp(n\pi H), n \in \mathbb{N}\}.$$

The quotient $F\backslash\mathbf{I}$ (*together with an appropriate metric*) *is a model of massive non-rotating BTZ black hole* (*see* [4]). *The bivector field it inherits following the above remark, is Poisson. Its symplectic leaves consist of the projection to* $F\backslash\mathbf{I}$ *of the orbits of the action of* G^σ *on* S *except along the projection of the orbit of the identity. Along this orbit, the bivector field vanishes and each point forms a symplectic leaf.*

In the coordinates (46) *of* [4] (*or* (4)*of the present article*), *the Poisson bivector field is*

$$2\cosh^2\left(\frac{\rho}{2}\right)\sin(\tau)\sinh(\rho)\partial_\tau \wedge \partial_\theta. \tag{2}$$

The above Poisson bivector field should be compared with the one defined in [4] and given by

$$\frac{1}{\cosh^2(\rho/2)\sin(\tau)}\partial_\tau \wedge \partial_\theta.$$

The symplectic leaves of this Poisson structure are the images under the projection $\mathbf{I} \longrightarrow \mathbf{F}\backslash\mathbf{I}$ of the action of G_+^σ on S. Considering the similarity between the above two Poisson structures, it would be interesting to find an interpretation of this similarity from the black hole point of view.

3. Let the computations begin

Throughout the present article, I will use the notations introduced in the previous Section. To begin with, I will prove that $(D, G_+^\sigma, \mathfrak{g}_-^\sigma)$ does indeed form a quasi-triple.

Because $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, one also has a decomposition $\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$. One also has $\mathfrak{d} = \mathfrak{g}_+^\sigma \oplus \mathfrak{g}_-^\sigma$ and accordingly $\mathfrak{d}^* = \mathfrak{g}_+^{\sigma\,*} \oplus \mathfrak{g}_-^{\sigma\,*}$. Denote $p_{\mathfrak{g}_+^\sigma}$ and $p_{\mathfrak{g}_-^\sigma}$ the projections on respectively $\mathfrak{g}_+^\sigma$ and $\mathfrak{g}_-^\sigma$ induced by the decomposition $\mathfrak{d} = \mathfrak{g}_+^\sigma \oplus \mathfrak{g}_-^\sigma$. So that $1_\mathfrak{d} = p_{\mathfrak{g}_+^\sigma} + p_{\mathfrak{g}_-^\sigma}$.

In this article, I express results using mostly the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$. Using it, we have

$$\mathfrak{g}_+^{\sigma\,*} = \{(\xi, \xi \circ \sigma) \mid \xi \in \mathfrak{g}^*\}$$

and

$$\mathfrak{g}_-^{\sigma\,*} = \{(\xi, -\xi \circ \sigma) \mid \xi \in \mathfrak{g}^*\}.$$

Proposition 3.1. *The triple $(D, G_+^\sigma, \mathfrak{g}_-^\sigma)$ forms a quasi-triple in the sense of [1]. The characteristic elements of this quasi-triple as defined in [1] and hereby denoted by j, $\mathbf{F}^\sigma$, φ^σ and the r-matrix $r_\mathfrak{d}^\sigma$ are*

$$\begin{aligned} j : \mathfrak{g}_+^{\sigma\,*} &\longrightarrow \mathfrak{g}_-^\sigma \\ (\xi, \xi \circ \sigma) &\longmapsto \Delta_-^\sigma \circ K^{-1}(\xi), \end{aligned}$$

and

$$F^\sigma = 0,$$

and

$$\begin{aligned} \varphi^\sigma : \textstyle\bigwedge^3 \mathfrak{g}_+^{\sigma\,*} &\longrightarrow \mathbb{R} \\ ((\xi, \sigma \circ \xi), (\eta, \sigma \circ \eta), (\nu, \sigma \circ \eta)) &\longmapsto 2K(K^{-1}(\nu), [K^{-1}(\xi), K^{-1}(\eta)]), \end{aligned}$$

and finally the r-matrix

$$\begin{aligned} r_\mathfrak{d}^\sigma : \mathfrak{g}^* \oplus \mathfrak{g}^* &\longrightarrow \mathfrak{g} \oplus \mathfrak{g} \\ (\xi, \eta) &\longmapsto \tfrac{1}{2}\Delta_-^\sigma \circ K^{-1}(\xi + \eta \circ \sigma). \end{aligned}$$

Notice here that it is crucial for σ to be of order no higher than 2, otherwise one would fail to obtain a quasi-triple as in the above Proposition.

Proof. It is straightforward to prove that $\mathfrak{d} = \mathfrak{g}_+^\sigma \oplus \mathfrak{g}_-^\sigma$ and that both $\mathfrak{g}_+^\sigma$ and $\mathfrak{g}_-^\sigma$ are isotropic in $(\mathfrak{d}, \langle\,\rangle)$. This proves that $(D, G^\sigma, \mathfrak{g}_-^\sigma)$ is a quasi-triple.

For $(\xi, \xi \circ \sigma)$ in $\mathfrak{g}_+^{\sigma\,*}$ and $(x, \sigma(x))$ in $\mathfrak{g}_+^\sigma$

$$\langle j(\xi, \xi \circ \sigma), (x, \sigma(x))\rangle = (\xi, \xi \circ \sigma)(x, \sigma(x)).$$

The map j is actually characterised by this last property. The equality

$$\langle \Delta_-^\sigma \circ K^{-1} \circ \Delta_+^{\sigma *}(\xi, \xi \circ \sigma), (x, \sigma(x))\rangle = (\xi, \xi \circ \sigma)(x, \sigma(x)),$$

proves that

$$j(\xi, \xi \circ \sigma) = \Delta_-^\sigma \circ K^{-1} \circ \Delta_+^{\sigma *}(\xi, \xi \circ \sigma) = \Delta_-^\sigma \circ K^{-1}(\xi).$$

Since σ is a Lie algebra morphism, we have $[\mathfrak{g}_-^\sigma, \mathfrak{g}_-^\sigma] \subset \mathfrak{g}_+^\sigma$. This proves that $F^\sigma : \bigwedge^2 \mathfrak{g}_+^{\sigma\,*} \longrightarrow \mathfrak{g}_-^\sigma$, given by

$$F^\sigma(\xi, \eta) = p_{\mathfrak{g}_-^\sigma}[j(\xi), j(\eta)],$$

vanishes.

I will now compute φ^σ. It is defined as

$$\begin{aligned}
\varphi^\sigma((\xi, \sigma \circ \xi), (\eta, \sigma \circ \eta), (\nu, \sigma \circ \nu)) &= (\nu, \sigma \circ \nu) \circ p_{\mathfrak{g}_+^\sigma}([j(\xi, \sigma \circ \xi), j(\eta, \sigma \circ \eta)]) \\
&= \langle j(\nu, \sigma \circ \nu), [j(\xi, \sigma \circ \xi), j(\eta, \eta \circ \eta)]\rangle \\
&= \langle \Delta_-^\sigma \circ K^{-1}(\nu), [\Delta_-^\sigma \circ K^{-1}(\xi), \Delta_-^\sigma \circ K^{-1}(\eta)]\rangle \\
&= 2K(K^{-1}(\nu), [K^{-1}(\xi), K^{-1}(\eta)]).
\end{aligned}$$

Finally, the r-matrix is defined as

$$\begin{aligned}
&r_{\mathfrak{d}}^\sigma : \mathfrak{g}_+^{\sigma\,*} \oplus \mathfrak{g}_-^{\sigma\,*} \longrightarrow \mathfrak{g}_+^\sigma \oplus \mathfrak{g}_-^\sigma \\
&((\xi, \xi \circ \sigma), (\eta, \eta \circ \sigma)) \longmapsto (0, j(\xi, \xi \circ \sigma)).
\end{aligned}$$

If (ξ, η) is in $\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$ then its decomposition in $\mathfrak{g}_+^{\sigma\,*} \oplus \mathfrak{g}_-^{\sigma\,*}$ is $(1/2)((\xi + \eta \circ \sigma, \xi \circ \sigma + \eta), (\xi - \eta \circ \sigma, -\xi \circ \sigma + \eta))$. The result follows. □

I now wish to compute the bivector P_D^σ on D. By definition, it is equal to $(r_{\mathfrak{d}}^\sigma)^\lambda - (r_{\mathfrak{d}}^\sigma)^\rho$, where the upper script λ means the left invariant section of $\Gamma(TD \otimes TD)$ generated by $r_{\mathfrak{d}}^\sigma$, while the upper script ρ means the right invariant section of $\Gamma(TD \otimes TD)$ generated by $r_{\mathfrak{d}}^\sigma$.

Proposition 3.2. *Identify $T_d D$ to $\mathfrak{d}$ by right translations. The value of P_D^σ at $d = (a, b)$ is*

$$\begin{aligned}
&\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^* \longrightarrow \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \\
&(\xi, \eta) \longmapsto \tfrac{1}{2}(K^{-1}(\eta \circ \sigma \circ (\mathrm{Ad}_{\sigma(b)a^{-1}} - 1)), -K^{-1}(\xi \circ \sigma \circ (\mathrm{Ad}_{\sigma(a)b^{-1}}))).
\end{aligned}$$

Proof. Fix $d = (a, b)$ in D. I choose to trivialise the tangent bundle, and its dual, of D by using right translations. See $(r_{\mathfrak{d}}^\sigma)^\rho$ as a map from T^*D to TD. If α is in $\mathfrak{d}^*$, then

$$(r_{\mathfrak{d}}^\sigma)^\rho(d)(\alpha^\rho) = (r_{\mathfrak{d}}^\sigma(\alpha))^\rho(d),$$

whereas

$$(r_{\mathfrak{d}}^{\sigma})^{\lambda}(d)(\alpha^{\rho}) = (\mathrm{Ad}_d \circ r_{\mathfrak{d}}^{\sigma}(\alpha \circ \mathrm{Ad}_d))^{\rho}(d).$$

Thus P_D^{σ} at the point $d = (a, b)$ is

$$\begin{aligned}&\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^* \longrightarrow \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}\\ &(\xi, \eta) \longmapsto -\tfrac{1}{2}\Delta_- \circ K^{-1}(\xi + \eta \circ \sigma) + \tfrac{1}{2}\mathrm{Ad}_d \circ \Delta_- \circ K^{-1}(\xi \circ \mathrm{Ad}_a + \eta \circ \mathrm{Ad}_b \circ \sigma).\end{aligned}$$

The above description of P_D^{σ} can be simplified:

$$\begin{aligned}P_D^{\sigma}(d)(\xi, \eta) &= -\tfrac{1}{2}\Delta_- \circ K^{-1}(\xi + \eta \circ \sigma) + \tfrac{1}{2}(\mathrm{Ad}_a \circ K^{-1}(\xi \circ \mathrm{Ad}_a + \eta \circ \mathrm{Ad}_b\sigma),\\ &\quad -\sigma \circ \mathrm{Ad}_{\sigma(b)} \circ K^{-1}(\xi \circ \mathrm{Ad}_a + \eta \circ \mathrm{Ad}_b \circ \sigma))\\ &= -\tfrac{1}{2}\Delta_- \circ K^{-1}(\xi + \eta \circ \sigma) + \tfrac{1}{2}(K^{-1}(\xi + \eta \circ \mathrm{Ad}_b\sigma \circ \mathrm{Ad}_{a^{-1}}),\\ &\quad -\sigma \circ K^{-1}(\xi \circ \mathrm{Ad}_{a\sigma(b)^{-1}} + \eta \circ \mathrm{Ad}_b \circ \sigma \circ \mathrm{Ad}_{\sigma(b)^{-1}}))\\ &= -\tfrac{1}{2}(K^{-1}(\xi + \eta \circ \sigma), -\sigma \circ K^{-1}(\xi + \eta \circ \sigma))\\ &\quad + \frac{1}{2}(K^{-1}(\xi + \eta \circ \sigma \circ \mathrm{Ad}_{\sigma(b)a^{-1}}), -\sigma \circ K^{-1}(\xi \circ \mathrm{Ad}_{a\sigma(b)^{-1}} + \eta \circ \sigma))\\ &= \tfrac{1}{2}(K^{-1}(\eta \circ \sigma \circ (\mathrm{Ad}_{\sigma(b)a^{-1}} - 1)), -\sigma \circ K^{-1}(\xi \circ (\mathrm{Ad}_{a\sigma(b)^{-1}} - 1)))\\ &= \tfrac{1}{2}(K^{-1}(\eta \circ \sigma \circ (\mathrm{Ad}_{\sigma(b)a^{-1}} - 1)), -K^{-1}(\xi \circ \sigma \circ (\mathrm{Ad}_{\sigma(a)b^{-1}} - 1))). \quad \square\end{aligned}$$

It follows from [1] that P_D^{σ} is projectable on $S^{\sigma} = D/G_+^{\sigma}$. Actually, the following is also true

Proposition 3.3. *The bivector P_D^{σ} is projectable to a bivector P_S^{σ} on $S = D/G_+$. Identify S with G through the map*

$$\begin{aligned}&D \longrightarrow G\\ &(a, b) \longmapsto ab^{-1}.\end{aligned}$$

Trivialise the tangent space to G, and hence to S, by right translations. If s is in S, then using the above identification P_S^{σ} at the point s is

$$P_S^{\sigma}(s)(\xi) = \tfrac{1}{2}(\mathrm{Ad}_{\sigma(s)^{-1}} - \mathrm{Ad}_s) \circ \sigma \circ K^{-1}(\xi). \tag{3}$$

Following a suggestion of the referee, one can define the semi-direct product

$$\tilde{G} = \mathbb{Z}_2 \ltimes G,$$

where the non-trivial element of $\mathbb{Z}_2$ acts on G as σ. If one identifies the component of the identity of $\tilde{G}$ to G_+^{σ} and the other component to S, then the action by conjugation of $\tilde{G}$

to itself restricts to the twisted action (1). This construction might explain the surprising observation that the bi-vector P_D^σ descends to S. Of course, one still needs to do some work here since, for example, one needs the groups D and G to be connected in order to apply the results of [1].

Proof. Assume s in S is the image of (a,b) in D, that is $s = ab^{-1}$. The tangent map of

$$\begin{aligned} &D \longrightarrow G \\ &(a, b) \longmapsto ab^{-1} \end{aligned}$$

at (a,b) is

$$\begin{aligned} &p : \mathfrak{d} \longrightarrow \mathfrak{g} \\ &(x, y) \longmapsto x - \mathrm{Ad}_{ab^{-1}} y. \end{aligned}$$

The dual map of p is

$$\begin{aligned} &p^* : \mathfrak{g}^* \longrightarrow \mathfrak{d}^* \\ &\xi \longmapsto (\xi, -\xi \circ \mathrm{Ad}_{ab^{-1}}) \end{aligned}$$

The bivector P_D^σ is projectable onto S if and only if for all (a,b) in D and ξ in $\mathfrak{g}^*$, the expression

$$p(P_D^\sigma(a, b)(p^*\xi))$$

depends only on $s = ab^{-1}$. It will then be equal to $P_S^\sigma(s)(\xi)$. This expression is equal to

$$\begin{aligned} p(P_D^\sigma(a, b)(\xi, -\xi \circ \mathrm{Ad}_{ab^{-1}})) &= \tfrac{1}{2} p((\mathrm{Ad}_{a\sigma(b)^{-1}} - 1) \circ \sigma \circ K^{-1}(-\xi \circ \mathrm{Ad}_{ab^{-1}}), \\ &\quad (1 - \mathrm{Ad}_{b\sigma(a)^{-1}}) \circ \sigma \circ K^{-1}(\xi)) \\ &= \tfrac{1}{2} p((\mathrm{Ad}_{\sigma(ba^{-1})} - \mathrm{Ad}_{a\sigma(a)^{-1}}) \circ \sigma \circ K^{-1}(\xi), \\ &\quad (1 - \mathrm{Ad}_{b\sigma(a)^{-1}}) \circ \sigma \circ K^{-1}(\xi)) \\ &= \tfrac{1}{2}(\mathrm{Ad}_{\sigma(ba^{-1})} - \mathrm{Ad}_{a\sigma(a)^{-1}} - \mathrm{Ad}_{ab^{-1}} \\ &\quad + \mathrm{Ad}_{a\sigma(a)^{-1}}) \circ \sigma \circ K^{-1}(\xi) \\ &= \tfrac{1}{2}(\mathrm{Ad}_{\sigma(ba^{-1})} - \mathrm{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi). \end{aligned}$$

This both proves that P_D^σ is projectable on S and gives a formula for the projected bivector. □

To prove that there exists a quasi-Poisson action of G^σ on (S, P_S^σ), I must compute $[P_S^\sigma, P_S^\sigma]$, where [,] is the Schouten-Nijenhuis bracket on multi-vector fields.

Lemma 3.4. *For x, y and z in $\mathfrak{g}$, let $\xi = K(x)$, $\eta = K(y)$ and $\nu = K(z)$. We have*

$$\frac{1}{2}[P_S^\sigma(s), P_S^\sigma(s)](\xi, \eta, \nu) = \frac{1}{4}K(x, [y, \tau_s(z)] + [\tau_s(y), z] - \tau_s([y, z])),$$

where $\tau_s = \mathrm{Ad}_s \circ \sigma - \sigma \circ \mathrm{Ad}_{s^{-1}}$.

Proof. Let (a, b) in D be such that $s = ab^{-1}$. Let p be as in the proof of Proposition 3.3. The bivector $P_S^\sigma(s)$ is $p(P_D^\sigma(a, b))$. Hence,

$$[P_S^\sigma(s), P_S^\sigma(s)] = p([P_D^\sigma(a, b), P_D^\sigma(a, b)]).$$

But it is proved in [1] that

$$[P_D^\sigma(a, b), P_D^\sigma(a, b)] = (\varphi^\sigma)^\rho(a, b) - (\varphi^\sigma)^\lambda(a, b).$$

Hence

$$\frac{1}{2}[P_S^\sigma(s), P_S^\sigma(s)] = p((\varphi^\sigma)^\rho(a, b)) - p((\varphi^\sigma)^\lambda(a, b)).$$

Now, it is tedious but straightforward and very similar to the above computations to check that

$$p((\varphi^\sigma)^\rho(a, b))(\xi, \eta, \nu) = \frac{1}{4}K(x, [y, \tau_s(z)] + [\tau_s(y), z] - \tau_s([y, z])),$$

and

$$p((\varphi^\sigma)^\lambda(a, b))(\xi, \eta, \nu) = 0. \quad \square$$

The group D acts on $S = D/G_+$ by multiplication on the left. This action restricts to an action of G_+^σ on S. Identifying G and G_+^σ via Δ_+^σ, this action is

$$\begin{aligned} G \times S &\longrightarrow S \\ (g, s) &\longmapsto gs\sigma(g)^{-1}. \end{aligned}$$

The infinitesimal action of $\mathfrak{g}$ at the point s in S reads

$$\begin{aligned} \mathfrak{g} &\longrightarrow T_sS \simeq \mathfrak{g} \\ x &\longmapsto x - \mathrm{Ad}_s \circ \sigma(x), \end{aligned}$$

with dual map

$$\begin{aligned} T_s^*S \simeq \mathfrak{g}^* &\longrightarrow \mathfrak{g}^* \\ \xi &\longmapsto \xi - \xi \circ \mathrm{Ad}_s \circ \sigma. \end{aligned}$$

Denote by $(\varphi^\sigma)_S$ the induced trivector field on S. If ξ, η and ν are in $\mathfrak{g}^*$ then

$$(\varphi^\sigma)_S(s)(\xi, \eta, \nu) = \varphi^\sigma(\xi - \xi \circ \mathrm{Ad}_s \circ \sigma, \eta - \eta \circ \mathrm{Ad}_s \circ \sigma, \nu - \nu \circ \mathrm{Ad}_s \circ \sigma).$$

Computing the right hand side in the above equality is a simple calculation which proves the following Lemma.

Lemma 3.5. *The bivector field P_S^σ and the trivector field $(\varphi^\sigma)_S$ satisfy*

$$\tfrac{1}{2}[P_S^\sigma, P_S^\sigma] = (\varphi^\sigma)_S.$$

To prove that the action of G_+^σ on (S, P_S^σ) is indeed quasi-Poisson, there only remains to prove that P_S^σ is G_+^σ-invariant.

Lemma 3.6. *The bivector field P_S^σ* is *G_+^σ-invariant.*

Proof. Fix G in $G \simeq G_+^\sigma$. Denote Σ_g the action of G on S. The tangent map of Σ_g at $s \in S$ is

$$\begin{aligned} T_sS \simeq \mathfrak{g} &\longrightarrow T_{gs\sigma(g)^{-1}}S \simeq \mathfrak{g} \\ x &\longmapsto \mathrm{Ad}_g x. \end{aligned}$$

Also, if ξ is in $\mathfrak{g}^*$

$$\begin{aligned} P_S^\sigma(gs\sigma(g)^{-1})(\xi) &= \tfrac{1}{2}(\mathrm{Ad}_{g\sigma(s)^{-1}\sigma(g)^{-1}} - \mathrm{Ad}_{gs\sigma(g)^{-1}}) \circ \sigma \circ K^{-1}(\xi) \\ &= \tfrac{1}{2}\mathrm{Ad}_g \circ (\mathrm{Ad}_{\sigma(s)^{-1}} - \mathrm{Ad}_s) \circ \mathrm{Ad}_{\sigma(g)^{-1}} \circ \sigma \circ K^{-1}(\xi) \\ &= \mathrm{Ad}_g(P_S^\sigma(s)(\xi \circ \mathrm{Ad}_g)) = (\Sigma_g)_*(P_S^\sigma)(\Sigma_g(s))(\xi). \quad \square \end{aligned}$$

Lemma 3.7. *Let s be in S. The image of $P_S^\sigma(s)$ is*

$$\mathrm{Im} P_S^\sigma(s) = \{(1 - \mathrm{Ad}_s \circ \sigma) \circ (1 + \mathrm{Ad}_s \circ \sigma)(y) | y \in \mathfrak{g}\}.$$

In particular, it is included in the tangent space to the orbit through s of the action of G^σ.

Proof. The image of $P_S^\sigma(s)$ is by Proposition 3.3

$$\mathrm{Im} P_S^\sigma(s) = \{(\mathrm{Ad}_{\sigma(s)^{-1}} - \mathrm{Ad}_s)\sigma(x) | x \in \mathfrak{g}\}.$$

The Lemma follows by setting $x = \mathrm{Ad}_s \circ \sigma(y) = \sigma \circ \mathrm{Ad}_{\sigma(s)}(y)$ and noticing that $(1 - (\mathrm{Ad}_s \circ \sigma)^2) = (1 - \mathrm{Ad}_s \circ \sigma) \circ (1 + \mathrm{Ad}_s \circ \sigma)$. $\square$

This finishes the proof of Theorem 2.1.

Choose G and σ as in Theorem 2.2. The trivector field $[P_S^\sigma, P_S^\sigma]$ is tangent to the orbit of the action of G_+^σ on S. These orbits are of dimension at most 2, therefore the trivector field $[P^\sigma_S, P^\sigma_S]$ vanishes and P^σ_S defines a Poisson structure on $\mathrm{SL}(2,\mathbb{R})$ which is invariant under the action

$$\begin{aligned}&\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\longrightarrow\mathrm{SL}(2,\mathbb{R})\\&(g,s)\longmapsto gs\sigma(g)^{-1}.\end{aligned}$$

Lemma 3.7 and a simple computation prove that along the orbit of the identity, the bivector field P^σ_S vanishes; and that elsewhere, its image coincides with the tangent space to the orbits of the above action. Recall that in [4], the domain **I** is given by

$$z(\tau,\theta,\rho)=\begin{bmatrix}\sinh(\frac{\rho}{2})+\cosh(\frac{\rho}{2})\cos(\tau) & \exp(\theta)\cosh(\frac{\rho}{2})\sin(\tau)\\ -\exp(-\theta)\cosh(\frac{\rho}{2})\sin(\tau) & -\sinh(\frac{\rho}{2})+\cosh(\frac{\rho}{2})\cos(\tau)\end{bmatrix}. \tag{4}$$

This formula also defines coordinates on I. Using Formula (3) of Proposition 3.3 and a computer, it is easy to check that P^σ_S if indeed given by Formula (2). This ends the proof of Theorem 2.2.

4. Final remarks

One might ask how different is the quasi-Poisson action of G^σ_+ on (S, P^σ_S) from the usual quasi-Poisson action of G_+ on (S, P_S). For example, if one takes $G = \mathrm{SU}(2)$, $H = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $\sigma = \mathrm{Ad}_H$ then the multiplication on the right in SU(2) by $\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$ defines an isomorphism between the two quasi-Poisson actions.

Nevertheless, in the example of Theorem 2.2, the two structures are indeed different since for example the action of $\mathrm{SL}(2,\mathbb{R})$ on itself by conjugation has two fixed points whereas the action of $\mathrm{SL}(2,\mathbb{R})$ on itself used in Theorem 2.2 does not have any fixed point.

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